Numerical Analysis of Nonlinear Dynamical Systems

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Abstract

Nonlinearities govern most observable phenomena which cannot be explained by a mathematical relationship of proportionality. The breakthrough in nonlinear dynamics happened in the 1800's when the French mathematician Henri Poincaré responded to the impossibility of exactly solving the *three body problem* by using a powerful geometric approach to analyze questions related to the stability and equilibrium of such dynamical systems. The same field later saw a boom with the invention of high speed computers and development of numerical methods. Despite being deeply mathematical, the science of nonlinear systems is applicable to almost everything starting from the weather prediction to chemical reactions. In this report we deal with two popular mechanical systems, namely *the double pendulum* to portray it's chaotic behavior using computational methods and *the bead on a rotating hoop* to demonstrate bifurcation and the power of linear stability analysis (LSA).

MOTIVATION

Newton solved the two-body problem using his laws of Gravitation but when the latter generation of scientists tried to apply Newton's methods to three-body problems, it was found too difficult to solve.

During the latter stages of the 19th century, Poincare tried to introduce a new approach to analyze the same problem giving emphasis on qualitative rather than quantitative aspects, in response to a prize declared by King Oscar II of Sweden and Norway on his 60th birthday. He was the first person to observe the possibility of chaos. He saw that in nonlinear systems, the system behavior was sensitive to initial conditions, thus it is difficult to make long-term predictions.

During the latter half of 20th-century development of high power computational systems, allowed scientists to play with equations, thus it was possible to develop some intuition about nonlinear systems.

MATHEMATICAL DEFINITIONS

What is meant by nonlinearity in the context of ordinary differential equations?

An ordinary differential equation of the form,

$$p_n(\frac{d^n y}{dx^n})^{a_n} + p_{n-1}(\frac{d^{n-1} y}{dx^{n-1}})^{a_{n-1}} + ... + p_1(\frac{dy}{dx})^{a_1} + p_0 y^{a_0} = f(x)$$

is nonlinear if either or both of the following is true,

i) any one of $a_n, a_{n-1}, ..., a_1, a_0$ is > 1
ii) any one of $p_n, p_{n-1}, ..., p_1, p_0$ is a function of the dependent variable y.

For example, the governing equation for a spring mass system where y is s i.e. displacement and x is t i.e. time, becomes nonlinear when either mass m or stiffness k is a function of dependent variable s.

$$m\ddot{s} + ks = g(t)$$

What is an equilibrium solution?

First, a solution of any equation satisfies the equation. The *equilibrium* solutions of an ordinary differential equation would be those for which all higher order rates of change of the dependent variable with respect to the independent variable go to zero i.e. y when $\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, ..., \frac{dy}{dx}$ are all zero.

Since most dynamical phenomena have *time* as the independent variable, it means that if the dependent variable (generalized coordinate) ϕ attains an equilibrium solution, it does not change with respect to time, without external efforts. In the presence of a forcing term $g(t)$ the equilibrium solutions would be time dependent, else they are constant with time.

What is meant by stability of the equilibrium solution?

If we choose a value of ϕ slightly different (perturbed) from the equilibrium solution, say ϕ_0, then some or all of the higher order derivatives must become non-zero to satisfy the equation causing it to evolve over time. In case ϕ remains close to the nearest equilibrium solution as time progresses to infinity, the solution is said to be *stable*. If it diverges away, it is *unstable*. A solution is *neutrally* stable if any arbitrary perturbed value in the neighborhood of the equilibrium solution is also an equilibrium solution.

Linear Stability Analysis.
How?

As is evident from the name, stability of a solution is analyzed after linearizing the nonlinear equation. Consider a small perturbation from the equilibrium solution ϕ_0 as ϕ'.
— Substitute ϕ by $\phi_0 + \phi'$ in the equation.
— Results in two parts, one comprising ϕ_0 which resembles the original equation and the other comprising perturbations ϕ'.
— The first part is removed as ϕ_0 satisfies the equation.
— Second and higher order terms of ϕ' are neglected as they are small compared to others.
— This results in a linear ordinary differential equation with ϕ' as the variable.
— Every equilibrium solution results in a different equation, which can be solved to understand the evolution of the local perturbation with time.

Why?

In most cases, nonlinear equations do not have analytical solutions and hence numerical methods are used to solve them. Linear Stability Analysis (LSA) allows us to have a qualitative understanding of a few important solutions of the equation and the behavior of the dependent variable in the neighborhood of such solutions.

What is bifurcation?

The change in properties of equilibrium solutions (appearance, stability) depending on the value of a particular constant (parameter) in the equation is called bifurcation. This is illustrated in the following example of a bead on a rotating hoop.

What is chaos?

Chaos is *aperiodic long-termed* behavior in a *deterministic* system that exhibits *sensitive dependence on initial conditions*. The double pendulum exhibits a dramatic demonstration of chaotic motion.

i) aperiodic long-termed behavior implies that the system never settles down into a stable configuration
ii) deterministic means that all possibilities of the irregular motion being due to noise or random input is ruled out, i.e. nonlinearity of the system is responsible
iii) sensitive dependence on initial conditions says that even if two initial conditions that are very close to each other, the result of each will be tremendously different.

APPENDIX
Bead on Rotating Hoop

The classical problem of a *bead on a rotating hoop* has been considered, to show bifurcation in a mechanical system and the power of *linear stability analysis*.

A bead of mass m is allowed to rotate freely on a hoop of radius r. The hoop rotates about a vertical diametric axis with a constant angular velocity ω. The angular position of the bead with respect to the downward vertical in the plane of rotation is θ.

The forces acting on the bead would be,
- *gravitational*
- *centrifugal*

The range of θ is taken to be $-\pi < \theta \leq \pi$ so that there is exactly one point on the hoop that corresponds to a particular angle. The perpendicular distance of the bead from the vertical axis is $\rho = r\sin(\theta)$. *Newton's laws of motion* or the *Euler-Lagrange equation* are used to obtain the equation of motion for the bead on the hoop. Thus,

$$mr\ddot{\theta} = -mg\sin(\theta) + mr\omega^2\sin(\theta)\cos(\theta)$$

This is a second order nonlinear equation in terms of θ. The equilibrium points (solutions) of this are obtained by setting the time derivatives of the variable θ to zero. These are given by,

$$\sin(\theta) = 0 \quad or \quad \Omega^2\cos(\theta) = 1$$

where $\Omega^2 = \omega^2 r/g$. This means θ is either $0, \pi$ or $\cos^{-1}(g/\omega^2 r)$. The last two equilibrium points exist only beyond a certain value of ω given by $\omega \geq \sqrt{g/r}$. In Fig. 2 the hollow dots represent unstable equilibria and the filled ones represent stable equilibria.

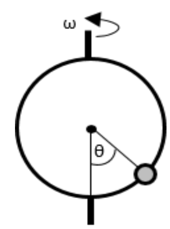

Figure 1: Schematic

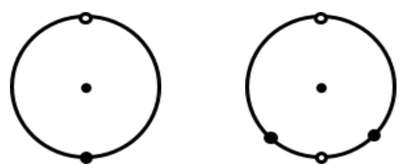

Figure 2: Stable and unstable equilibrium solutions for Ω^2 less than and greater than 1 respectively

Now *linear stability analysis* is carried out to understand the nature of the equilibrium solutions. For this, a small perturbation θ' is given to the equilibrium solution assumed to be $\bar{\theta}$ *i.e.*

$$\theta = \bar{\theta} + \theta'$$

Substituting these back in the previous equation and keeping the following in mind, we get,

- *orders of perturbations higher than 1 are neglected*
- *the cosine of a perturbation is 1 and sine is equal to itself*
- *the derivative of barred quantities is zero as they are constants (solutions)*

$$\ddot{\theta} = \frac{g}{r}(\Omega^2\cos(\theta) - 1)\sin(\theta)$$

$$or \quad \ddot{\bar{\theta}} + \ddot{\theta}' = \frac{g}{r}(\Omega^2\cos(\bar{\theta} + \theta') - 1)\sin(\bar{\theta} + \theta')$$

Expanding the sine and cosine terms,

$$or \quad \ddot{\theta}' = \frac{g}{r}[\Omega^2\cos(\bar{\theta})\cos(\theta') - \Omega^2\sin(\bar{\theta})\sin(\theta') - 1][\sin(\bar{\theta})\cos(\theta') + \cos(\bar{\theta})\sin(\theta')]$$

$$or \quad \ddot{\theta'} = \frac{g}{r}[\Omega^2 \cos(\bar{\theta}) - \Omega^2 \sin(\bar{\theta})\theta' - 1][\sin(\bar{\theta}) + \cos(\bar{\theta})\theta']$$

By substituting the different equilibrium solutions in place of $\bar{\theta}$, the type of equilibrium is determined.

NOTE: Solution to $\ddot{x} = -p^2x$, is of the form $Ae^{-ipt} + Be^{+ipy}$ or $C\cos(pt) + D\sin(pt)$ which is bounded at all time (stable). Solution to $\ddot{x} = +p^2x$, is of the form $Ae^{-pt} + Be^{+pt}$, of which the former goes to zero and the later goes to infinity with time (unstable).

Case 1: $\bar{\theta} = 0$

$$\ddot{\theta'} = \frac{g}{r}[\Omega^2 - 1]\theta'$$

This is a second order equation of the form $\ddot{x} = \pm p^2x$ depending on the value of Ω^2 compared to 1. The *solution is unstable* if $\Omega^2 \geq 1$ ($\ddot{x} = +p^2x$). The *solution is stable* if $\Omega^2 < 1$ ($\ddot{x} = -p^2x$).

Case 2: $\bar{\theta} = +\pi \ \ or \ -\pi$

$$\ddot{\theta'} = \frac{g}{r}[\Omega^2 + 1]\theta'$$

which is of the form ($\ddot{x} = +p^2x$) and thus the *solution is unstable* for all values of Ω.

Case 3: $\bar{\theta} = \pm|\cos^{-1}(1/\Omega^2)|$
Substituting sine in terms of Ω in the expression for $\ddot{\theta'}$,

$$\ddot{\theta'} = -\frac{g}{\Omega^2 r}[(\Omega^4 - 1)\theta' + \theta'^2\sqrt{\Omega^4 - 1}]$$

Neglecting higher powers of θ',

$$\ddot{\theta'} = -\frac{g}{\Omega^2 r}[(\Omega^4 - 1)\theta']$$

As mentioned earlier, for the solution $\bar{\theta} = \pm|\cos^{-1}(1/\Omega^2)|$ to exist, Ω^2 must be greater than 1 which makes this an equation of the form $\ddot{x} = -p^2x$, rendering the solution stable due to reasons described earlier.

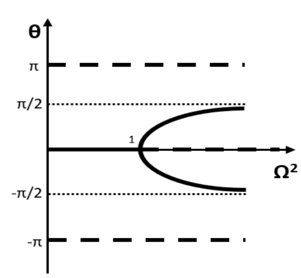

Figure 3: Bifurcation diagram

This is an example of Pitchfork Bifurcation. The dotted and solid lines represent unstable and stable equilibria respectively.

Chaos in Double Pendulum

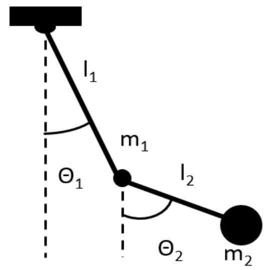

Figure 4: Double pendulum schematic diagram

Derivation of equations of motion

$$x_1 = l_1 \sin\theta_1 \quad \textit{and} \quad y_1 = -l_1 \cos\theta_1$$
$$x_2 = l_2 \sin\theta_2 + l_1 \sin\theta_1 \quad \textit{and} \quad y_2 = -(l_2 \cos\theta_2 + l_1 \cos\theta_1)$$
$$\dot{x_1} = l_1\dot{\theta_1}\cos\theta_1 \quad \textit{and} \quad \dot{y_1} = l_1\dot{\theta_1}\sin\theta_1$$
$$\dot{x_2} = l_1\dot{\theta_1}\cos\theta_1 + l_2\dot{\theta_2}\cos\theta_2 \quad \textit{and} \quad \dot{y_2} = l_1\dot{\theta_1}\sin\theta_1 + l_2\dot{\theta_2}\sin\theta_2$$

Find T(Kinetic energy) and V(Potential energy),

$$V = m_1 g_1 y_1 + m_2 g_2 y_2$$

$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$
$$T = 0.5 m_1 {v_1}^2 + 0.5 m_2 {v_2}^2$$
$$T = 0.5 m_1 {l_1}^2 \dot{\theta_1}^2 + 0.5 m_2({l_1}^2\dot{\theta_1}^2 + {l_2}^2\dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2))$$

The Euler-Lagrange Equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \left(\frac{\partial L}{\partial q_i}\right) = Q_i$$

where q_i = Generalised Coordinates , Q_i = Generalised Forces
and $L = T - V$, Lagrangian

$$L = 0.5(m_1+m_2){l_1}^2\dot{\theta_1}^2 + 0.5 m_2 {l_2}^2 \dot{\theta_2}^2 + m_2 l_1 l_2 \dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2) + (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

$$\frac{\partial L}{\partial \dot{q_i}} = \frac{\partial L}{\partial \dot{\theta_1}} = (m_1 + m_2){l_1}^2\dot{\theta_1} + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta_2}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_1}}) = (m_1{+}m_2){l_1}^2\ddot{\theta_1}{+}m_2 l_1 l_2 \ddot{\theta_2}\cos{(\theta_1-\theta_2)}{-}m_2 l_1 l_2 \dot{\theta_2}(\dot{\theta_1}{-}\dot{\theta_2})\sin{(\theta_1-\theta_2)}$$

$$(\frac{\partial L}{\partial \theta_1}) = -l_1 g_1 (m_1+m_2)\sin\theta_1 - m_2 l_1 l_2 \dot{\theta_1}\dot{\theta_2}\sin(\theta_1-\theta_2)$$

$Q_i = 0$ (No generalised forces)
From the above, we get the first equation of motion as,

$$(m_1{+}m_2)l_1\ddot{\theta_1}{+}m_2 l_2\ddot{\theta_2}\cos(\theta_1{-}\theta_2){-}m_2 l_2{\dot{\theta_2}}^2\sin(\theta_1{-}\theta_2){+}(m_1{+}m_2)g\sin\theta_1 = 0$$

Similarly, the second equation of motion is given as,

$$m_2 l_2\ddot{\theta_2} + m_2 l_1\ddot{\theta_1}\cos(\theta_1-\theta_2) - m_2 l_1{\dot{\theta_1}}^2\sin(\theta_1-\theta_2) + m_2 g\sin\theta_2 = 0$$

So,

$$\omega_1 = \ddot{\theta_1} = \frac{-g(2m_1+m_2)\sin\theta_1 - m_2 g\sin{(\theta_1-2\theta_2)} - 2m_2({\theta_2}^2 l_2 + {\dot{\theta_1}}^2\cos(\theta_1-\theta_2))\sin(\theta_1-\theta_2)}{l_1(2m_1+m_2-m_2\cos{(2\theta_1-2\theta_2)}}$$

$$\omega_2 = \ddot{\theta_2} = \frac{-g(2m_1+m_2)\sin\theta_1 - m_2 g\sin{(\theta_1-2\theta_2)} - 2m_2({\theta_2}^2 l_2 + {\dot{\theta_1}}^2\cos(\theta_1-\theta_2))\sin(\theta_1-\theta_2)}{l_2(2m_1+m_2-m_2\cos{(2\theta_1-2\theta_2)}}$$

The sensitive dependence of motion of double pendulum on initial condition is demonstrated below through images by considering two identical double pendula released with slightly different initial condition. A Forward Euler time discretization MATLAB code was written for this purpose.

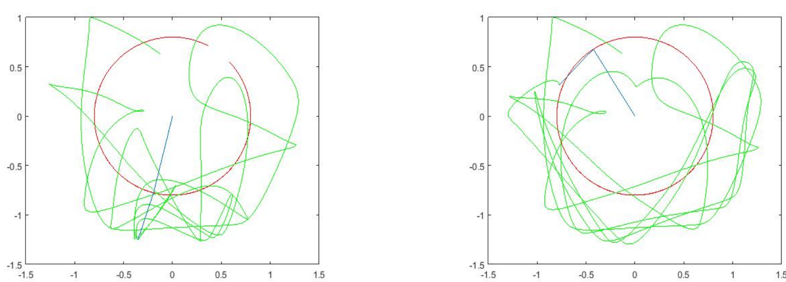

Figure 5: Chaotic responses of two identical double pendula released from initial condition $\theta_1 = 2.64, \theta_2 = -1.4$ and $\theta_1 = 2.64, \theta_2 = -1.402$

Any difference in initial conditions, however tiny, is sufficient to make a overwhelming difference in motion of the double pendulum.

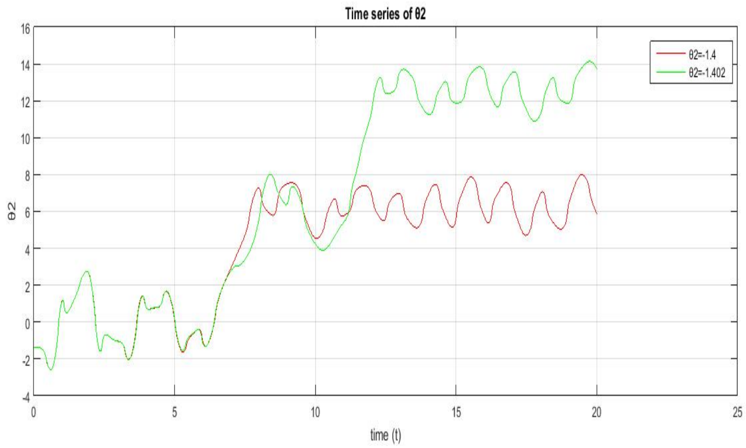

Figure 6: Time evolution of θ_2

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1. Steven H. Strogatz. *Nonlinear Dynamics and Chaos.*
2. https://math.stackexchange.com/